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Deconstructing scalar QED at zero and finite temperature

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Abstract. We calculate the effective potential for the WLPNGB in a world with a circular latticized extra dimension. The mass of the Wilson line pseudo-Nambu–Goldstone boson (WLPNGB) is calculated from the one-loop quantum effect of scalar fields at zero and finite temperature. We show that a series expansion by the modified Bessel functions is useful to calculate the one-loop effective potentials.

1 Introduction

It seems obvious that we live in a four-dimensional world. Nevertheless, many theories for unification of forces and/or matter in more dimensions than four have been studied [1]. A simple possibility is that there is a fifth dimension of very tiny size attached to every point of our four-dimensional world. Such an extra dimension can hardly be seen by virtue of its extraordinary smallness.

Last year, there appeared a novel scheme to describe higher-dimensional gauge theories, called "deconstruction" [2,3]. A number of copies of a four-dimensional theory linked by new fields can be viewed as a single gauge theory. The resulting whole theory may be almost equivalent to a higher-dimensional theory with discretized extra dimensions.

Recently, Hill and Leibovich pointed out that the Wilson line pseudo-Nambu–Goldstone boson (WLPNGB) with low mass can be naturally obtained by deconstructing five-dimensional QED [4,5]. This WLPNGB may be an important candidate for cosmological quintessence.

For a cosmological application, we should take the finite-temperature effect into account. The behavior of the WLPNGB field may vary along with the cosmological evolution.

In this paper, we examine the $U(1)$ gauge theory with a discretized circle. We analytically obtain the effective potential for the WLPNGB at zero and finite temperature. In this paper, we consider the one-loop effect of charged scalar bosons. Although this model appears unnatural in contrast to the model with fermions [4,5], a similar technique is valid for the other models and the application to various models will be studied elsewhere.

In Sect. 2, our model is explained and the mass spectra of the component fields are shown. In Sect. 3, the one-loop quantum effect of scalar fields is calculated at zero temperature. In Sect. 4, the one-loop quantum effect of scalar fields is calculated at finite temperature. The final section, Sect. 5, is devoted to our conclusions.

2 Model

We begin with the lagrangian for deconstructing $(d+1+1)$-D scalar QED:

$$\begin{aligned}\mathcal{L} = &\sum_{k=1}^{N}\frac{1}{g^2}\left[-\frac{1}{4}F_k^{\mu\nu}F_{k\ \mu\nu} - (D^\mu U_k)^\dagger D_\mu U_k\right]\\ &+\sum_{k=1}^{N}\left[-(D^\mu\tilde{\phi}_k)^\dagger D_\mu\tilde{\phi}_k\right]\\ &+ f\sum_{k=1}^{N}\left(\sqrt{2}\tilde{\phi}_k^* U_k\tilde{\phi}_{k+1} + \sqrt{2}\tilde{\phi}_k^* U_{k-1}^*\tilde{\phi}_{k-1} - 2f\tilde{\phi}_k^*\tilde{\phi}_k\right)\\ &- m^2\sum_{k=1}^{N}\tilde{\phi}_k^*\tilde{\phi}_k\,,\end{aligned} \tag{1}$$

where

$$F_k^{\mu\nu} = \partial^\mu\tilde{A}_k^\nu - \partial^\nu\tilde{A}_k^\mu\,,\quad D^\mu\tilde{\phi}_k = \partial^\mu\tilde{\phi}_k - \mathrm{i}\tilde{A}_k^\mu\tilde{\phi}_k \tag{2}$$

and

$$D^\mu U_k = \partial^\mu U_k - \mathrm{i}\tilde{A}_k^\mu U_k + \mathrm{i}U_k\tilde{A}_{k+1}^\mu\,. \tag{3}$$

The labels of the fields are considered as periodic modulo N, e.g., $\phi_{N+1}\equiv\phi_1$, $\phi_0\equiv\phi_N$, and so on. N is assumed to be larger than $(d+1)/2$. Usually, the dimension of the *space* is taken as three. We leave the dimensions unfixed because of the possibility of some combination of compactification schemes in the very early universe.

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We assume that all U_k have a common absolute value $|U_k| = f/\sqrt{2}$. Hence we can write

$$U_k = \frac{f}{\sqrt{2}} \exp\left(\mathrm{i}\tilde{\chi}_k/f\right) . \quad (4)$$

It is convenient to use the "Fourier transformed" modes for the fields:

$$\tilde{A}^\mu_k = \frac{1}{\sqrt{N}} \sum_p A^\mu_p \exp\left[2\pi\mathrm{i}\frac{pk}{N}\right] ,$$
$$\tilde{\phi}_k = \frac{1}{\sqrt{N}} \sum_p \phi_p \exp\left[2\pi\mathrm{i}\frac{pk}{N}\right] , \quad (5)$$

and

$$\tilde{\chi}_k = \frac{1}{\sqrt{N}} \sum_p \chi_p \exp\left[2\pi\mathrm{i}\frac{pk}{N}\right] . \quad (6)$$

The fields A^μ_p $(p \neq 0)$ acquire masses by "absorbing" the χ_p $(p \neq 0)$; the mass spectrum is given by

$$4f^2 \sin^2\left(\frac{\pi p}{N}\right) . \quad (7)$$

For small p, this mass spectrum is approximately given by

$$f^2 \left(\frac{2\pi p}{N}\right)^2 , \quad (8)$$

which is the Kaluza–Klein spectrum in the continuum theory with the circle of the circumference $L = N/f$.

The masses of charged bosons are

$$M_p^2 = 4f^2 \sin^2\left(\frac{\pi p}{N} + \frac{\bar{\chi}}{2f}\right) + m^2 , \quad (9)$$

where $\bar{\chi} \equiv \chi_0/\sqrt{N}$ is a (classically) zero-mode scalar field.

3 The effective potential at zero temperature

3.1 The one-loop effective potential

In this section, we compute the quantum effect of the scalar fields at zero temperature. The one-loop effective potential for $\bar{\chi}$ is obtained by

$$\ln\det[-\nabla^2 + M_p^2(\bar{\chi})] \quad (10)$$
$$\sim -\frac{1}{(2\pi)^{d+1}} \sum_p \int_0^\infty \frac{\mathrm{d}t}{t} \int \mathrm{d}^{d+1}\mathbf{k}\ \exp\left[-(\mathbf{k}^2 + M_p^2)t\right]$$
$$= -\frac{1}{(4\pi)^{(d+1)/2}} \int_0^\infty \frac{\mathrm{d}t}{t} t^{-(d+1)/2} \sum_p \exp\left[-M_p^2 t\right] ,$$

after an appropriate regularization.

Using the formula

$$\exp\left[-4f^2\sin^2(\theta/2)t\right] = \mathrm{e}^{-2f^2t} \sum_{\ell=-\infty}^{\infty} \cos\ell\theta\ I_\ell(2f^2t)$$
$$= \mathrm{e}^{-2f^2t} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\ell\theta} I_\ell(2f^2t) , \quad (11)$$

where $I_\nu(x)$ is the modified Bessel function, we can write the effective potential as

$$V_0(\bar{\chi}) = -\frac{2}{(4\pi)^{(d+1)/2}} \sum_p \sum_{\ell=1}^{\infty} \cos\ell\theta_p\ \mathcal{I}(\ell; m) , \quad (12)$$

where

$$\theta_p \equiv \frac{2\pi p}{N} + \frac{\bar{\chi}}{f} , \quad (13)$$

and

$$\mathcal{I}(\ell; m) = \int_0^\infty \frac{\mathrm{d}t}{t^{(d+3)/2}} \mathrm{e}^{-(2f^2+m^2)t} I_\ell(2f^2t) . \quad (14)$$

Here the term which is independent of $\bar{\chi}$ is discarded.

Carrying out the summation over p first, we find that only the term of $p = qN$ (q is an integer) is left. Then we find

$$V_0(\bar{\chi}) = -\frac{2N}{(4\pi)^{(d+1)/2}} \sum_{q=1}^{\infty} \cos(qN\bar{\chi}/f)\ \mathcal{I}(qN; m) . \quad (15)$$

3.2 $m = 0$

First, we examine the case of $m = 0$ in detail. One can find [6]

$$\mathcal{I}(qN; 0) = (4f^2)^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{d+2}{2}\right) \Gamma\left(qN - \frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(qN + \frac{d+1}{2} + 1\right)} . \quad (16)$$

Therefore the effective potential for the WLPNGB is written as

$$V_0(\bar{\chi}) = -\frac{2N\Gamma\left(\frac{d+2}{2}\right) f^{d+1}}{\pi^{(d+2)/2}} \times \sum_{q=1}^{\infty} \frac{\Gamma\left(qN - \frac{d+1}{2}\right)}{\Gamma\left(qN + \frac{d+1}{2} + 1\right)} \cos(qN\bar{\chi}/f) . \quad (17)$$

In particular, when $d = 3$, we obtain

$$V_0(\bar{\chi}) = -\frac{3f^4}{2\pi^2} \sum_{q=1}^{\infty} \frac{\cos(qN\bar{\chi}/f)}{q(q^2N^2-1)(q^2N^2-4)} . \quad (18)$$

Turning back to the case with general d, we find that the effective potential for a large N can be expressed as

$$V_0(\bar{\chi}) = -\frac{2\Gamma\left(\frac{d+2}{2}\right)}{\pi^{(d+2)/2}L^{d+1}} \left[\sum_{q=1}^{\infty} \frac{\cos(qL\bar{\chi})}{q^{d+2}} + \frac{(d+1)(d+2)(d+3)}{24N^2} \sum_{q=1}^{\infty} \frac{\cos(qL\bar{\chi})}{q^{d+4}} + O(N^{-4})\right] , \quad (19)$$

where $L \equiv N/f$.

The mass of the χ_0 field is derived from the effective potential and turns out to be

$$m_\chi^2 = \frac{2\Gamma\left(\frac{d+2}{2}\right) \tilde{g}^2}{\pi^{(d+2)/2}L^{d-1}} Z(d, N) , \quad (20)$$

with

$$
\begin{aligned}
Z(d,N) &\equiv \sum_{q=1}^{\infty} \frac{q^2 N^{d+2} \Gamma\left(qN - \frac{d+1}{2}\right)}{\Gamma\left(qN + \frac{d+1}{2} + 1\right)} \\
&= \zeta(d) + \frac{(d+1)(d+2)(d+3)}{24N^2}\zeta(d+2) \\
&\quad + O(N^{-4}) \,, \qquad (21)
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta function and $\tilde{g} = g/\sqrt{N}$. Note that the kinetic term of χ_0 includes the factor $\tilde{g}^{-2}$.

$Z(3,N)$ and $Z(2,N)$ are plotted against N in Fig. 1. It is safe to say that the approximation of the "large N limit" is very good even for $N \approx 10$.

3.3 $m \gg f$

For finite m, (14) reduces to [6]

$$
\begin{aligned}
\mathcal{I}(qN;m) &= (2f^2+m^2)^{\frac{d+1}{2}} \left(\frac{f^2}{2f^2+m^2}\right)^{qN} \\
&\times \frac{\Gamma\left(qN - \frac{d+1}{2}\right)}{\Gamma(qN+1)} \\
&\times {}_2F_1\left(\frac{qN - \frac{d+1}{2}}{2}, \frac{qN - \frac{d-1}{2}}{2}; qN+1; \frac{4f^4}{(2f^2+m^2)^2}\right) \\
&= m^{d+1}\left(\frac{f^2}{m^2}\right)^{qN} \frac{\Gamma\left(qN - \frac{d+1}{2}\right)}{\Gamma(qN+1)} \\
&\times {}_2F_1\left(qN - \frac{d+1}{2}, qN + \frac{1}{2}; 2qN+1; -\frac{4f^2}{m^2}\right) \,, \qquad (22)
\end{aligned}
$$

where ${}_2F_1$ is the Gauss hypergeometric function.

In the large N limit, $\mathcal{I}(qN;m)$ behaves as $\mathrm{e}^{-mqN/f}$ for large m. For finite N, however, $\mathcal{I}(qN;m)$ approaches zero not exponentially but with a power law. We find that when $m \gg f$ (22) reduces to

$$
\mathcal{I}(qN;m) \sim m^{d+1}\left(\frac{f^2}{m^2}\right)^{qN} \frac{\Gamma\left(qN - \frac{d+1}{2}\right)}{\Gamma(qN+1)} \quad (m \gg f) \,. \qquad (23)
$$

Thus we obtain

$$
V_0(\bar{\chi}) \sim -\frac{2m^{d+1}}{(4\pi)^{(d+1)/2}}\left(\frac{f^2}{m^2}\right)^{N} \frac{\Gamma\left(N - \frac{d+1}{2}\right)}{\Gamma(N)} \cos(N\bar{\chi}/f) \quad (m \gg f) \,. \qquad (24)
$$

Correspondingly, the mass of the χ_0 field reads

$$
m_\chi^2 \sim \frac{2\tilde{g}^2 m^{d-1}}{(4\pi)^{(d+1)/2}}\left(\frac{f^2}{m^2}\right)^{N-1} \frac{N^2\Gamma\left(N - \frac{d+1}{2}\right)}{\Gamma(N)} \quad (m \gg f) \,, \qquad (25)
$$

which can be a very small value if we choose an appropriate value for f/m. This fact suggests that the model can bring about interesting models in a cosmological application.

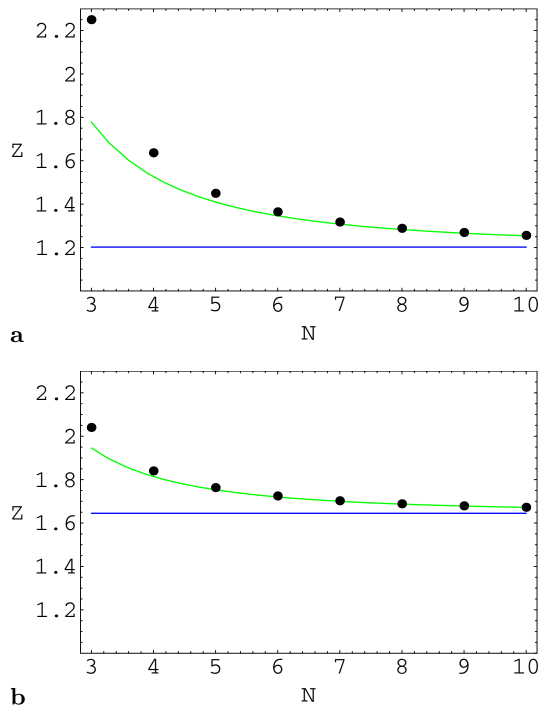

Fig. 1. **a** $Z(3,N)$ is plotted against N. The horizontal line indicates $\zeta(3)$. The curve illustrates the approximated value up to the order N^{-2}. **b** $Z(2,N)$ is plotted against N. The horizontal line indicates $\zeta(2)$. The curve illustrates the approximated value up to the order N^{-2}

4 The effective potential at finite temperature

4.1 The finite-temperature effective potential

We know that to study the finite-temperature effective potential, the integration over the frequency is replaced by the summation over the discrete Matsubara frequencies (and attaching a certain factor) [7]. The free energy density is then obtained:

$$
\begin{aligned}
F &= -\frac{1}{(2\pi)^d\beta}\sum_p \sum_{n'=-\infty}^{\infty} \int_0^\infty \frac{\mathrm{d}t}{t} \int \mathrm{d}^d\mathbf{k} \\
&\quad \times \exp\left\{-\left[\left(\frac{2\pi}{\beta}\right)^2 n'^2 + \mathbf{k}^2 + M_p^2\right]t\right\} \\
&= -\frac{1}{(4\pi)^{(d+1)/2}} \qquad (26) \\
&\quad \times \int_0^\infty \frac{\mathrm{d}t}{t} t^{-(d+1)/2} \sum_p \sum_{n=-\infty}^{\infty} \exp\left[-M_p^2 t - \frac{\beta^2 n^2}{4t}\right] \,,
\end{aligned}
$$

where $T = \beta^{-1}$ is the temperature. Obviously, the $n = 0$ term in the summation gives the effective potential at zero temperature.

Now we write F in the form

$$
F = V_0(\bar{\chi}) + \Delta V(\bar{\chi}) + F_0 \,. \qquad (27)
$$

Performing the summation over p, one can see that the finite-temperature correction to the potential $\Delta V(\bar{\chi})$ results in

$$\Delta V(\bar{\chi}) = -\frac{4N}{(4\pi)^{(d+1)/2}} \sum_{q=1}^{\infty} \cos\left[qN\frac{\bar{\chi}}{f}\right] \mathcal{T}(qN;m)\,, \quad (28)$$

and

$$F_0 = -\frac{2N}{(4\pi)^{(d+1)/2}}\, \mathcal{T}(0;m)\,, \quad (29)$$

where

$$\mathcal{T}(\ell;m) = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\mathrm{d}t}{t^{(d+3)/2}} \times \exp\left[-(2f^2+m^2)t - \frac{\beta^2 n^2}{4t}\right] I_\ell(2f^2 t)\,. \quad (30)$$

Expanding the modified Bessel functions, we can carry out the integration and obtain

$$\mathcal{T}(qN;m) = 2(2f^2+m^2)^{(d+1)/2}(f^2/(2f^2+m^2))^{qN} \times \sum_{r=0}^{\infty} \frac{(f^2/(2f^2+m^2))^{2r}}{r!\Gamma(qN+r+1)} \times \sum_{n=1}^{\infty} \left(\frac{\sqrt{2f^2+m^2}\beta n}{2}\right)^{2r+qN-\frac{d+1}{2}} \times K_{2r+qN-\frac{d+1}{2}}(\sqrt{2f^2+m^2}\beta n)\,, \quad (31)$$

where $K_\nu(z)$ is the McDonald function (or a modified Bessel function of the second type).

4.2 The high-temperature limit

In the high-temperature limit $\beta \to 0$, the summation over n can be replaced by integration and the following approximation is obtained:

$$\mathcal{T}(qN;m) \sim \frac{\sqrt{\pi}}{\beta}(2f^2+m^2)^{\frac{d}{2}} \times \left(\frac{f^2}{2f^2+m^2}\right)^{qN} \frac{\Gamma\left(qN-\frac{d}{2}\right)}{\Gamma(qN+1)} \times {}_2F_1\left(\frac{qN-\frac{d}{2}}{2}, \frac{qN-\frac{d}{2}+1}{2}; qN+1; \frac{4f^4}{(2f^2+m^2)^2}\right)$$
$$= \frac{\sqrt{\pi}}{\beta} m^d \left(\frac{f^2}{m^2}\right)^{qN} \frac{\Gamma\left(qN-\frac{d}{2}\right)}{\Gamma(qN+1)} \times {}_2F_1\left(qN-\frac{d}{2}, qN+\frac{1}{2}; 2qN+1; -\frac{4f^2}{m^2}\right) \quad (\beta^{-1} \gg f, m)\,. \quad (32)$$

There occurs nothing but the so-called dimensional reduction phenomenon in high-temperature field theory.

In the case of $m=0$, the high-temperature limit leads to

$$\mathcal{T}(qN;0) \sim \frac{1}{\beta}(4f^2)^{\frac{d}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(qN-\frac{d}{2}\right)}{\Gamma\left(qN+\frac{d}{2}+1\right)} \quad (\beta^{-1} \gg f)\,, \quad (33)$$

and the effective potential becomes

$$V(\bar{\chi}) \equiv V_0(\bar{\chi}) + \Delta V(\bar{\chi}) \sim -\frac{2N\Gamma\left(\frac{d+1}{2}\right) f^d}{\beta \pi^{(d+1)/2}} \sum_{q=1}^{\infty} \frac{\Gamma\left(qN-\frac{d}{2}\right)}{\Gamma\left(qN+\frac{d}{2}+1\right)} \cos(qN\bar{\chi}/f) \quad (\beta^{-1} \gg f)\,. \quad (34)$$

Particularly, for $d=3$,

$$V(\bar{\chi}) \sim -\frac{2f^3}{\beta\pi^2} \sum_{q=1}^{\infty} \frac{N\cos(qN\bar{\chi}/f)}{(q^2N^2-1/4)(q^2N^2-9/4)} \quad (\beta^{-1} \gg f) \quad (35)$$

is obtained. For general d and large N, we find

$$V(\bar{\chi}) \sim -\frac{2\Gamma\left(\frac{d+1}{2}\right)}{\beta\pi^{(d+1)/2}L^d} \times \left[\sum_{q=1}^{\infty} \frac{\cos(qL\bar{\chi})}{q^{d+1}} + \frac{d(d+1)(d+2)}{24N^2} \sum_{q=1}^{\infty} \frac{\cos(qL\bar{\chi})}{q^{d+3}} + O(N^{-4})\right] \quad (\beta^{-1} \gg f)\,, \quad (36)$$

where $L \equiv N/f$.

The mass of the χ_0 field in the high-temperature limit is

$$m_\chi^2 \sim \frac{2\Gamma\left(\frac{d+1}{2}\right)\tilde{g}^2}{\beta\pi^{(d+1)/2}L^{d-2}} Z(d-1,N) \quad (\beta^{-1} \gg f)\,, \quad (37)$$

where $Z(d,N)$ has been defined as (21).

In the case that $\beta^{-1} \gg m \gg f$, we find

$$\mathcal{T}(qN;m) \sim \frac{\sqrt{\pi}}{\beta} m^d \left(\frac{f^2}{m^2}\right)^{qN} \frac{\Gamma\left(qN-\frac{d}{2}\right)}{\Gamma(qN+1)} \quad (\beta^{-1} \gg m \gg f)\,, \quad (38)$$

and this leads to

$$V(\bar{\chi}) \sim -\frac{2m^d}{\beta(4\pi)^{d/2}} \left(\frac{f^2}{m^2}\right)^{N} \frac{\Gamma\left(N-\frac{d}{2}\right)}{\Gamma(N)} \cos(N\bar{\chi}/f) \quad (\beta^{-1} \gg m \gg f)\,. \quad (39)$$

The mass of the χ_0 field is then

$$m_\chi^2 \sim \frac{2\tilde{g}^2 m^{d-2}}{\beta(4\pi)^{d/2}} \left(\frac{f^2}{m^2}\right)^{N-1} \frac{N^2\Gamma\left(N-\frac{d}{2}\right)}{\Gamma(N)} \quad (\beta^{-1} \gg m \gg f)\,. \quad (40)$$

The mass-square of the χ_0 linearly increases with temperature.

4.3 Temperature dependence of the free energy

In the rest of this section, we investigate the leading temperature dependence of the free energy. Though the contribution of the gauge fields is of course present, we concentrate ourselves only on the contribution of the scalar fields.

The dominant dependence on temperature can be found in F_0. Let us remember

$$\begin{aligned}\mathcal{T}(0;m) &= 2(2f^2+m^2)^{(d+1)/2} \\ &\times \sum_{r=0}^{\infty}\frac{(f^2/(2f^2+m^2))^{2r}}{r!\Gamma(r+1)}\sum_{n=1}^{\infty}\left(\frac{\sqrt{2f^2+m^2}\beta n}{2}\right)^{2r-\frac{d+1}{2}} \\ &\times K_{2r-\frac{d+1}{2}}(\sqrt{2f^2+m^2}\beta n)\,,\end{aligned} \tag{41}$$

where we should notice that $K_\nu(z)=K_{-\nu}(z)$.

At extremely high temperature ($\beta^{-1}\gg f, m$), the $r=0$ term is dominant, and, using the limiting form for a small argument $K_\nu(z)\sim\frac{1}{2}\Gamma(|\nu|)(z/2)^{-|\nu|}$, one obtains

$$\mathcal{T}(0;m)\sim\frac{2^{d+1}\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\beta^{d+1}}\quad(\beta^{-1}\gg f,m)\,. \tag{42}$$

Then this leads to

$$F\sim F_0\sim-\frac{2N\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{(d+1)/2}\beta^{d+1}}\quad(\beta^{-1}\gg f,m)\,. \tag{43}$$

This is precisely the free energy for N (effectively) massless charged bosons. This behavior can be derived from the original form of $\mathcal{T}(0;m)$:

$$\begin{aligned}\mathcal{T}(0;m) &= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{\mathrm{d}t}{t^{(d+3)/2}} \\ &\times\exp\left[-(2f^2+m^2)t-\frac{\beta^2n^2}{4t}\right]I_0(2f^2t) \\ &= \frac{1}{\beta^{d+1}}\sum_{n=1}^{\infty}\frac{1}{n^{d+1}}\int_0^{\infty}\frac{\mathrm{d}t}{t^{(d+3)/2}} \\ &\times\exp\left[-(2f^2+m^2)\beta^2n^2t-\frac{1}{4t}\right]I_0(2f^2\beta^2n^2t)\,,\end{aligned} \tag{44}$$

with the limiting form $I_0(z)\sim 1$ for a small argument.

On the other hand, for $\beta^{-1}\ll f$, (44) can be approximated, using $I_0(z)\sim \mathrm{e}^z/\sqrt{2\pi z}$ for a large argument, by

$$\mathcal{T}(0;m)\sim\frac{2}{\sqrt{4\pi}\ f\beta^{d+2}}\sum_{n=1}^{\infty}\left(\frac{2\beta m}{n}\right)^{\frac{d+2}{2}}K_{\frac{d+2}{2}}(\beta mn) \quad (\beta^{-1}\ll f)\,. \tag{45}$$

This leads to

$$F\sim F_0\sim-\frac{4N}{(4\pi)^{(d+2)/2}\ f\beta^{d+2}}\times\sum_{n=1}^{\infty}\left(\frac{2\beta m}{n}\right)^{\frac{d+2}{2}}K_{\frac{d+2}{2}}(\beta mn) \quad (\beta^{-1}\ll f)\,. \tag{46}$$

Further, if we assume $m\ll\beta^{-1}$, it is found that

$$\mathcal{T}(0;m)\sim\frac{2^{d+2}\Gamma\left(\frac{d+2}{2}\right)\zeta(d+2)}{\sqrt{4\pi}\ f\beta^{d+2}} \quad (m\ll\beta^{-1}\ll f)\,. \tag{47}$$

Then in this case,

$$F\sim F_0\sim-\frac{2\Gamma\left(\frac{d+2}{2}\right)\zeta(d+2)}{\pi^{(d+2)/2}\beta^{d+2}}\frac{N}{f} \quad (m\ll\beta^{-1}\ll f) \tag{48}$$

is obtained. This result coincides with the one of the finite-temperature continuum Kaluza–Klein theory with circle length $L=N/f$ [8], after replacing the scalar degree of freedom.

We have found that $(-F)$ behaves as T^{d+1} at high temperature, while it behaves as T^{d+2} at lower temperature than f. This fact indicates that the dimension of the spacetime seems $d+2$ for $T<f$ and again $d+1$ for $T>f$. Of course, at very low temperature $T\ll f/N$, as one can see from (26) for $m=0$ and $\bar{\chi}=0$,

$$F\sim-\frac{2\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{(d+1)/2}\beta^{d+1}}\quad(\beta^{-1}\ll f/N)\,. \tag{49}$$

So we recognize the world as $(d+1)$-dimensional spacetime with the lowest-mode field at very low temperature.

5 Conclusion

In conclusion, the effective potential for the WLPNGB in scalar QED with a discretized dimension has been calculated at zero and finite temperature. We have utilized the expansion in terms of the modified Bessel functions, which is also useful for computing the one-loop effect in models with more discretized (or, latticized) dimensions [9]. We have found that approximating the one-loop effect by a large N expansion is valid if the model has a limiting form of infinite N.

We have also found that the $(\text{mass})^2$ of the WLPNGB increases linearly with high temperature. This serves to lead to some possibilities: Coherent oscillations of the WLPNGB field may change the frequency according to the expansion of the universe, or, if domain walls might be produced, their mass density decreases as temperature decreases. Furthermore, the novel temperature dependence of energy density may bring about interesting consequences to the very early universe. These cosmological implications will be clarified after analyzing more realistic models and incorporating other matter fields.

We should consider the one-loop effect of fermions for more natural particle theory. Moreover degenerate fermions may largely affect the WLPNGB mass and the entire potential. These subjects will be discussed elsewhere [10].

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